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HOMOGENIZATION OF PLAIN WEAVE COMPOSITES USING TWO-SCALE CONVERGENCE

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Abstract—In this paper, we homogenize a plain weave composite using convergence results developed for periodic functions. The *two-scale* convergence scheme is discussed and its results are applied to the elasticity problem of modeling composite weaves. Numerical implementation is discussed and parameter studies are given.

1. INTRODUCTION

A classical problem in mechanics and mathematics consists of obtaining equations which describe the global and/or averaged behavior of certain quantities of interest varying over domains which exhibit greatly heterogeneous microstructure. Often comprised of complex and uncharacterizable geometries, these domains typically necessitate approximations and/or premises which render the results questionable. A rigorous treatment, however, seems feasible for a certain class of microstructures and conditions, and it is this class that we choose to study. Microstructures in this set are described by highly oscillatory functions whose variations need not be perfectly uniform over the entire domain of interest; furthermore, the measure of the microstructural geometry is small when compared to particular global lengths. Such problems seem to have broad and practical applications, since they encompass a wide variety of structural solids such as composite materials, and so these are the problems we *homogenize*.

The mathematical process which leads to the theoretical simplification of the governing system of equations is termed *homogenization*, since the resulting systems contain only averaged-functions free of microstructurally influenced oscillations, and hence model homogeneous domains. What is indeed quite remarkable is that the solutions yielded by these systems are in fact the limit of the solutions of the original equations as the domain tends towards a homogeneous state. A completely rigorous mathematical theory which describes this process is presented in Section 2.

The power and practicality of the aforementioned homogenization scheme is demonstrated in Section 3. For this purpose, we consider the problem of determining the three-dimensional global elastic moduli of a composite material which consists of a matrix reinforced by a periodic array of plain weave fabrics. (This is the most common weave construction.) Therefore, we obtain a homogenization of this composite and state convergence results; namely, in the case of elasticity, not only do the displacements of the original problem converge to the displacements of the homogenized one, but so too do the stresses. This is in sharp contrast to the majority of engineering-type analyses of composite weaves (Pandey and Hahn, 1992; Dasgupta and Bhandarkar, 1994), crude approximations of the three-dimensional problem which essentially have no rigorous justification.

Finally, in Section 4, we present numerical results for some typical weave geometries. Parametric studies are performed, pertaining to the influence of fabric geometry on the global material properties.

2. THEORETICAL DEVELOPMENT

2.1. *Mathematical formulation*

The problem of modeling media which possess periodic structure is often accomplished by the analysis of elliptic differential operators A^ε taking the form

$$A^\varepsilon u^\varepsilon \equiv -\frac{\partial}{\partial x_i}\left[a^\varepsilon_{ij}\frac{\partial u^\varepsilon}{\partial x_j}\right] = f, \tag{1}$$

where eqn (1) is defined on an open set $\Omega \subset \mathbb{R}^N$, and either boundary conditions are prescribed on Ω or $\Omega = \mathbb{R}^N$. The coefficients a^ε_{ij} are highly oscillatory periodic functions on $\mathbb{R}^N$ which cause great difficulties in obtaining numerical solutions to eqn (1). For this reason, it is often desirable to *homogenize* this class of operators.

By this we mean that we wish to consider the solution u_0 of $Au_0 = f$, where A is the limit of the sequence A^ε as $\varepsilon \to 0$ in some appropriate topology. It is well known that if the sequence u^ε lies in some Banach space U, then $u^\varepsilon \to u_0$ weakly in U [see Žhikov *et al.* (1979); Babuska (1976); Bensoussan *et al.* (1978)]; however, finding the appropriate topology in which the operators $A^\varepsilon \to A$ is not a simple task and is the essence of Tartar's energy method. This method is used in classical homogenization schemes to validate the powerful, yet formal, asymptotic expansions in the method of multiple scales, yet it is often difficult to obtain the necessary test functions. When obtained, various modes of convergence are combined to give the homogenized operator A. However, the periodicity of the coefficients a^ε_{ij} is not fully exploited in the energy scheme even though the following necessary result is used; periodic functions in $L^\infty(\Omega)$ converge to their period average in the weak star topology as the period tends to zero.

First, after discussing the preliminaries and notation, we give an alternate proof from Ball and Murat (1984) for the above mentioned simple result, and show how it extends to functions $\psi(x, y)$, measurable and essentially bounded in x and continuous and periodic in y. Indeed, these are the essential ingredients of the "two-scale" convergence method first developed by Nguetseng (1989) and continued by Allaire (1992).

Then, we discuss this method, which is the mathematically rigorous version of the method of multiple scales, the difference being that in one step, both homogenization and convergence results are achieved (this method does fully utilize the periodicity of the coefficients). The asymptotic series ansatz for u^ε, which is *a priori* unknown to hold, is no longer required; moreover, it is not necessary to solve a hierarchical system of partial differential equations. Rather, Temam's (1979) theorem stating that gradients are orthogonal to divergence-free functions is used to find the limit to ∇u^ε in the two-scale topology which also furnishes the limit of u^ε. This, in turn, provides us with a homogenization of the original problem.

2.2. *Weak star convergence of periodic* L^∞*-functions*

We denote N-dimensional Euclidean space by $\mathbb{R}^N$, the natural numbers by $\mathbb{N}$, and the integers by $\mathbb{Z}$. Let Ω be a bounded Lipschitz domain, open in the measure space $(\mathbb{R}^N, \mathscr{L}, m)$ where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable sets and m is the Lebesgue measure on $\mathbb{R}^N$. For $1 \leqslant p \leqslant \infty$ we denote the equivalence class

$$L^p(\Omega, \mathscr{L}, m) = \{f : \Omega \to \mathbb{R} : f \text{measurable}, \|f\|_p < \infty\}$$

where

$$\|f\|_p = \left(\int_\Omega |f|^p\,\mathrm{d}x\right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \underset{x\in\Omega}{\text{ess sup}}|f(x)|,$$

with the equivalence relation $f \sim g$ iff $f = g$ a.e. We denote the Sobolev space $H^j(\Omega, \mathscr{L}, m)$ $(H^j_0(\Omega))$ to be the completion of $C^\infty(\Omega)$ $(C^\infty_0(\Omega))$ with respect to $\|\cdot\|_{H^j}$, where

$$\|f\|_{H^j}^2 = \int_\Omega \sum_{|\alpha| \leq j} |D^\alpha f|^2 \, dx.$$

Let $Y = [0, 1]^N$. We consider functions which are Y-periodic in $\mathbb{R}^N$. We define

$$E_j^\varepsilon = \varepsilon Y + \varepsilon j, \quad j \in \mathscr{Z}^N,$$

and denote by $C_{\text{per}}^\infty(Y)$, the space of infinitely differentiable functions on $\mathbb{R}^N$ that are Y-periodic, and so $L_{\text{per}}^p(Y)$ $(H_{\text{per}}^p(Y))$ is the completion of $C_{\text{per}}^\infty(Y)$ in the $L^p(Y)$ $(H^p(Y))$ norm. We note that elements of $L^p(\Omega, C_{\text{per}}(Y))$ take their values in the separable Banach space of continuous functions which are Y-periodic in y, and have norm

$$\|f\|_{L^p(\Omega, C_{\text{per}}(Y))}^p = \int_\Omega \sup_{y \in Y} |f(x,y)|^p \, dx.$$

We also note that $\sup_{y \in Y} |f(x,y)|$ is measurable on Ω if $f(x, \cdot)$ is continuous. Finally, we denote the characteristic function on $E \subset \mathbb{R}^N$ by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \in E^c$.

Theorem 1. Let $a(x/\varepsilon)$ be bounded in $L^\infty(\Omega)$, Y-periodic and let $a^\varepsilon(x) = a(x/\varepsilon)$. Then $a^\varepsilon \to \int_Y a \, dy$ in $L^\infty(\Omega)$ weak star as $\varepsilon \to 0$.

Proof. Let $U \in \mathscr{L}$ such that $m(U) < \infty$. By the regularity of the Lebesgue measure, for all $\delta > 0$, there exists a compact set K and an open set O such that $m(O) - \delta/2 < m(U) < m(K) + \delta/2$. Then $O = \{\cup_{\alpha \in A} V_\alpha : V_\alpha \text{ open rectangle in } \mathbb{R}^N\}$ (A arbitrary set) and $K \subset \cup_{l=1}^n V_l \subset O$ where $\{1, \ldots, n\} \subset A$, such that $|m(U) - m(\cup_{l=1}^n V_l)| < \delta$. For each V_l, there exists some $0 < \varepsilon_l \leq 1$ and some finite subset $F_l \subset \mathscr{Z}^N$ so that $V_l = \cup_{j \in F_l} E_j^{\varepsilon_l}$. Let $\varepsilon = \Pi_{l=1}^n \varepsilon_l$ and $F = \cup_{l=1}^n F_l$. Then

$$G^\varepsilon \equiv \cup_{j \in F} E_j^\varepsilon = \cup_{l=1}^n V_l, \quad \text{and} \quad \left| \int_U a^\varepsilon \, dx - \int_{G^\varepsilon} a^\varepsilon \, dx \right| \leq \delta \|a^\varepsilon\|_\infty.$$

Now $\{a^\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^\infty(\Omega)$, we can extract a subsequence that weak star converges. We show that the limit is indeed $\int_Y a \, dy$, and since it is independent of ε, the entire sequence converges. Since by a linear transformation

$$\int_{G^\varepsilon} a^\varepsilon \, dx = \sum_{j \in F} \int_{E_j^1} am(E_j^\varepsilon) \, dy,$$

we have by translation invariance of the measure, linearity of the integral, and periodicity of a

$$\sum_{j \in F} \int_{E_j^1} am(E_j^\varepsilon) \, dy = \int_Y \sum_{j \in F} a(y+j) m(E_j^\varepsilon) \, dy = \int_Y am(G^\varepsilon) \, dy.$$

Hence $|\int_\Omega a^\varepsilon 1_U \, dx - \int_Y am(U) \, dy| \leq 2\delta \|a^\varepsilon\|_\infty$, which combined with Fubini's theorem, and the density of simple functions in L^1 gives the desired result. □

Remark 1. If we consider the bilinear form $B^\varepsilon : V \times V \to \mathbb{R}$ defined by

$$B^\varepsilon(u^\varepsilon, v) = \int_\Omega a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx$$

where $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and the classical multi-scale asymptotic approach wherein

$$u^{\varepsilon}(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

then

$$B^{\varepsilon}(u^{\varepsilon}, v) = \int_{\Omega}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_j} - a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \chi^k}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k}\frac{\partial v}{\partial x_j}\right]\mathrm{d}x - O(\varepsilon)$$

[see Bensonssan *et al.* (1978), Sanchez-Palencia (1980)]. By Theorem 1, we see that

$$\lim_{\varepsilon \to 0} B^{\varepsilon}(u^{\varepsilon}, v) = \int_{\Omega}\int_{Y}\left[a_{ij}(y) - \frac{\partial \chi^i}{\partial y_k}a_{ik}(y)\right]\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_j}\mathrm{d}x$$

which is the classical homogenization result normally obtained by imposing the solvability condition for u_2.

Remark 2. We now extend the above mentioned result to periodic functions of the type $\phi(x, x/\varepsilon)$. Measurability of such functions is insured by imposing continuity on one of the two variables, so, for example, we consider functions in $L^p(\Omega, C_{\mathrm{per}}(Y))$ [Allaire (1992)]. Moreover, it is well known that the integrand of $\|\phi\|_{L^1(\Omega, C_{\mathrm{per}}(Y))}, \sup_{y \in Y}|\phi(x,y)|$ is measurable on Ω, for since ϕ is continuous in y, the supremum may be taken over the countable set of rational numbers $\mathcal{Q} \cap Y$; hence

$$(\sup_{y \in Y \cap \mathcal{Q}} |\phi(x,y)|)^{-1}(B) = \cup_{y \in \mathcal{Q} \cup Y}|\phi(x,y)|^{-1}(B)$$

for all Borel sets $B \subset \mathbb{R}$. Clearly, even if $\phi(x, \cdot)$ is a simple function having a finite number of discontinuities, the inverse image of the supremum is still a countable union of measurable sets. Functions in $L^p(\Omega, C_{\mathrm{per}}(Y))$ will be used as test functions in the topology of two-scale convergence. We note, however, that test functions in $\mathcal{D} \otimes C_{\mathrm{per}}(Y)$ can be used with a density argument, as well [see Nguetseng (1989)].

Corollary 1. If $\phi \in L^p(\Omega, C_{\mathrm{per}}(Y))$, then for $1 \leqslant p < \infty$

$$\lim_{\varepsilon \to 0}\left\|\phi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^p(\Omega)} = \|\phi(x,y)\|_{L^p(\Omega \times Y)}. \tag{2}$$

Proof. It is enough to consider $p = 1$, and by the triangle inequality, to show that

$$\theta \equiv \left|\int_{\Omega}\phi\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x - \int_{\Omega}\int_{Y}\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x\right| \to 0.$$

Let $\{\phi_n(x, \cdot)\}$ be a sequence of simple functions on Y *uniformly* converging to $\phi(x, \cdot)$. Then

$$\theta \leqslant \left|\int_{\Omega}\left(\phi\left(x, \frac{x}{\varepsilon}\right) - \phi_n\left(x, \frac{x}{\varepsilon}\right)\right)\mathrm{d}x\right| + \left|\int_{\Omega}\phi_n\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x - \int_{\Omega}\int_{Y}\phi_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\right| + \left|\int_{\Omega}\int_{Y}\phi_n(x,y) - \phi(x,y)\,\mathrm{d}y\,\mathrm{d}x\right|.$$

By extending the indicator functions in ϕ_n periodically in $\mathbb{R}^N$, the second term goes to zero as $\varepsilon \to 0$ by Theorem 1. Hence

$$\lim_{\varepsilon \to 0} \theta \leqslant 2 \int_\Omega \sup_{y \in Y} |\phi_n(x, y) - \phi(x, y)| \, \mathrm{d}x.$$

Since the integrand is measurable and bounded by $2 \sup_{y \in Y} |\phi(x, y)| \in L^1(\Omega)$, by dominated convergence $\theta \to 0$ as $n \to 0$. □

Clearly, with Ω bounded, letting ϕ be continuous in x and p-integrable in y does not change the proof and so functions in $C(\bar{\Omega}, L^p_{\mathrm{per}}(Y))$ also satisfy eqn (2). We note that functions in $C(\bar{\Omega}, L^\infty_{\mathrm{per}}(Y))$ also are shown to satisfy eqn (2) in [Allaire (1992)]. These type of functions are considered in Bensoussan *et al.* (1978).

2.3. *Homogenization by two-scale convergence*

We now present the theorem on the two-scale convergence of u^ε to u_0, first proven by Nguetseng (1989) and later simplified by Allaire (1992), but first we give its definition. Let $\{u^\varepsilon\}_{\varepsilon > 0} \subset L^2(\Omega)$. Then u^ε two-scale converges to $u_0 \in L^2(\Omega \times Y)$ ($u^\varepsilon \to u_0$), if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, \mathrm{d}y \, \mathrm{d}x \quad \forall \psi \in L^2(\Omega, C_{\mathrm{per}}(Y)). \tag{3}$$

Theorem 2. Let $\{u^\varepsilon\}_{\varepsilon > 0}$ be bounded in $L^2(\Omega)$. Then there exists a subsequence (still denoted by ε) and $u_0 \in L^2(\Omega \times Y)$ such that $u^\varepsilon \xrightarrow{2} u_0$.

Proof. We sketch Allaire's proof. For $\psi \in L^2(\Omega, C_{\mathrm{per}}(Y))$ and some $c > 0$, we have

$$\left| \int_\Omega u^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \right| \leqslant c \left\| \psi\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leqslant c \| \psi(x, y) \|_{L^2(\Omega, C_{\mathrm{per}}(Y))}$$

so by the Riesz representation theorem there exists $\{\mu_\varepsilon\}_{\varepsilon > 0} \subset L^2(\Omega, M_{\mathrm{per}}(Y))$ and $\mu_0 \in L^2(\Omega, M_{\mathrm{per}}(Y))$ so that $\mu_\varepsilon(\psi) \to \mu_0(\psi)$ up to a subsequence. Then as $\varepsilon \to 0$, $| < \mu_0, \psi > | \leqslant c \| \psi(x, y) \|_{L^2(\Omega \times Y)}$ by Corollary 1, so by the density of $L^2(\Omega, C_{\mathrm{per}}(Y))$ in $L^2(\Omega \times Y)$ and by the Riesz representation theorem, there exists $u_0 \in L^2(\Omega \times Y)$ satisfying $< \mu_0, \psi > = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, \mathrm{d}y \, \mathrm{d}x$. □

Let us consider the following second-order elliptic equation:

$$-\frac{\partial}{\partial x_i}\left[a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial x_j}\right] = f \quad \text{in} \quad \Omega \tag{4a}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \tag{4b}$$

where $\Omega \subset \mathbb{R}^N$, $f \in L^2(\Omega)$ and $a_{ij}(x, y)$ is bounded in x, Y-periodic in y and coercive, i.e. there exists $0 < \alpha \leqslant \beta$ such that

$$\alpha |\zeta|^2 \leqslant a_{ij}(x, y) \zeta_i \zeta_j \leqslant \beta |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^N. \tag{5}$$

We require $a_{ij}(x, y)$ to be measurable and satisfy

$$\lim_{\varepsilon \to 0} \int_\Omega a_{ij}\left(x, \frac{x}{\varepsilon}\right)^2 \mathrm{d}x = \int_\Omega \int_Y a_{ij}(x, y)^2 \, \mathrm{d}y \, \mathrm{d}x. \tag{6}$$

Hence, we note that the solution of eqn (4) satisfies the following variational problem:

$$u^\varepsilon \in H_0^1(\Omega)$$

$$\int_\Omega a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega),$$

and letting $v = u^\varepsilon$ [using eqn (5)] we see that $\| u^\varepsilon \|_{H_0^1(\Omega)} < c$ for all $\varepsilon > 0$. Thus, there exists $u_0 \in H_0^1(\Omega)$ such that $u^\varepsilon \to u_0$ weakly in $H_0^1(\Omega)$ up to a subsequence. By using Theorem 2 and the well known result that the orthogonal of divergence-free functions are exactly the gradients [see Temam (1979)], the following result is obtained [see Nguetseng (1989) or Allaire (1992) for proof]:

Lemma 1. Let $\{u^\varepsilon\}_{\varepsilon > 0} \subset H^1(\Omega)$ such that u^ε converges weakly to a limit u_0 in $H^1(\Omega)$. Then $u^\varepsilon \xrightarrow{2} u_0$ and there exists $u_1(x, y) \in L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$† such that $\nabla u^\varepsilon \xrightarrow{2} \nabla_x u_0(x) + \nabla_y u_1(x, y)$.

With this lemma, we are now able to obtain the homogenized operator A of the sequence

$$\left\{ -\frac{\partial}{\partial x_i} \left[a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \right] \right\}_{\varepsilon > 0}$$

of eqn (4). We state this result in the form of a theorem whose proof is based on Theorem 2.

Theorem 3. The solutions u^ε of eqn (4) converge weakly to $u_0(x)$ in $H_0^1(\Omega)$ and $\nabla u^\varepsilon \xrightarrow{2} \nabla_x u_0(x) + \nabla_y u_1(x, y)$, where (u_0, u_1) is the unique solution in $H_0^1(\Omega) \times L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$ of the following two-scale homogenized system‡:

$$-\frac{\partial}{\partial y_i}\left[a_{ij}(x, y) \frac{\partial u_0}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x, y) \right] = 0 \quad \text{in} \quad \Omega \times Y$$

$$-\frac{\partial}{\partial x_i}\left[\int_Y a_{ij}(x, y) \left(\frac{\partial u_0}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x, y) \right) \mathrm{d}y \right] = f \quad \text{in} \quad \Omega$$

$$u_0(x) = 0 \quad \text{on} \quad \partial\Omega$$

$$u_1(x, \cdot) \quad \text{is} \quad 1-\text{periodic}. \tag{7}$$

It is evident that the two-scale homogenized problem is a solution of two equations (or two sets of equations in elasticity, for example) with the two unknowns u_0 and u_1, each dependent on both the macroscopic variable x and the microscopic variable y. For the simple case of second-order elliptic equations, a decoupling occurs via the relation

$$u_1(x, y) = \frac{\partial u_0}{\partial x_i}(x) \, \chi_i(x, y); \tag{8}$$

however, in general a relation as this does not exist. Moreover, it is evident from eqn (3) that the limiting element u_0 does depend on the micro-coordinate y. Nevertheless, in all cases where $u^\varepsilon \to u_0$ weakly in $H^1(\Omega)$, the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ gives us $u^\varepsilon \to u_0$ in $L^2(\Omega)$ strongly and so the asymptotic approximation u_0 is independent of the microscopic variable y.

† $H^1_{\text{per}}(Y)/\mathbb{R}$ is the closed subspace of $H^1_{\text{per}}(Y)$ defined by $\{f \in H^1_{\text{per}}(Y) : \int_Y f \, \mathrm{d}y = 0\}$. We let $\| f \|^2_{H^1_{\text{per}}(Y)/\mathbb{R}} = \sum_{|\alpha| = 1} \| D^\alpha f \|^2_{L^2(Y)}$.

‡This is the term used by Allaire (1992) to define the system of partial differential equations which must be solved simultaneously for the macro and micro variables.

3. APPLICATION TO MULTI-LAYERED PLAIN WEAVE COMPOSITES

In this section, we consider the homogenization of a three-dimensional multi-layered weave composite, comprised of two orthogonal sets of yarns embedded in resin (Fig. 1). The microstructure of this material is periodic in all three directions, and Fig. 2 illustrates the period cell. We wish to analyse the elastic response of a body Ω, bounded and connected in $\mathbb{R}^3$, comprised of just such a material, and study the behavior of the solution as the periodicity tends to zero of the following system of equations of linear elasticity:

$$-\frac{\partial}{\partial x_j}\left[C_{ijkl}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_k^\varepsilon}{\partial x_l}\right]=f_i \quad \text{in} \quad \Omega \tag{9a}$$

$$u^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon,u_3^\varepsilon)=0 \quad \text{on} \quad \partial\Omega \tag{9b}$$

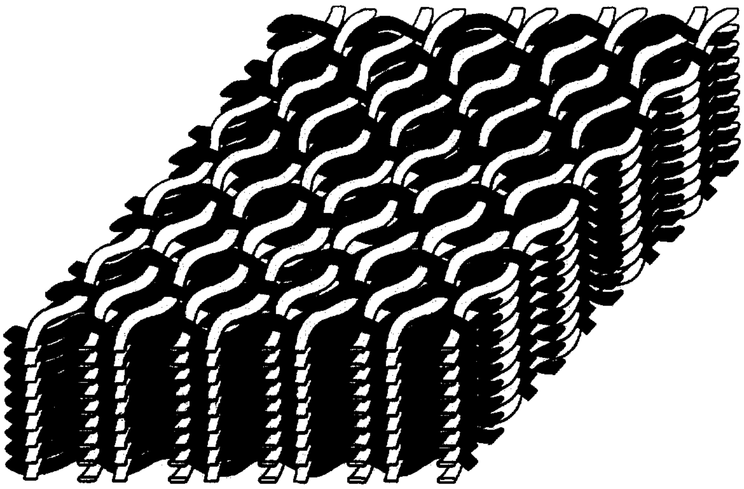


Fig. 1. Multi-layered plain weave yarns in composite material.

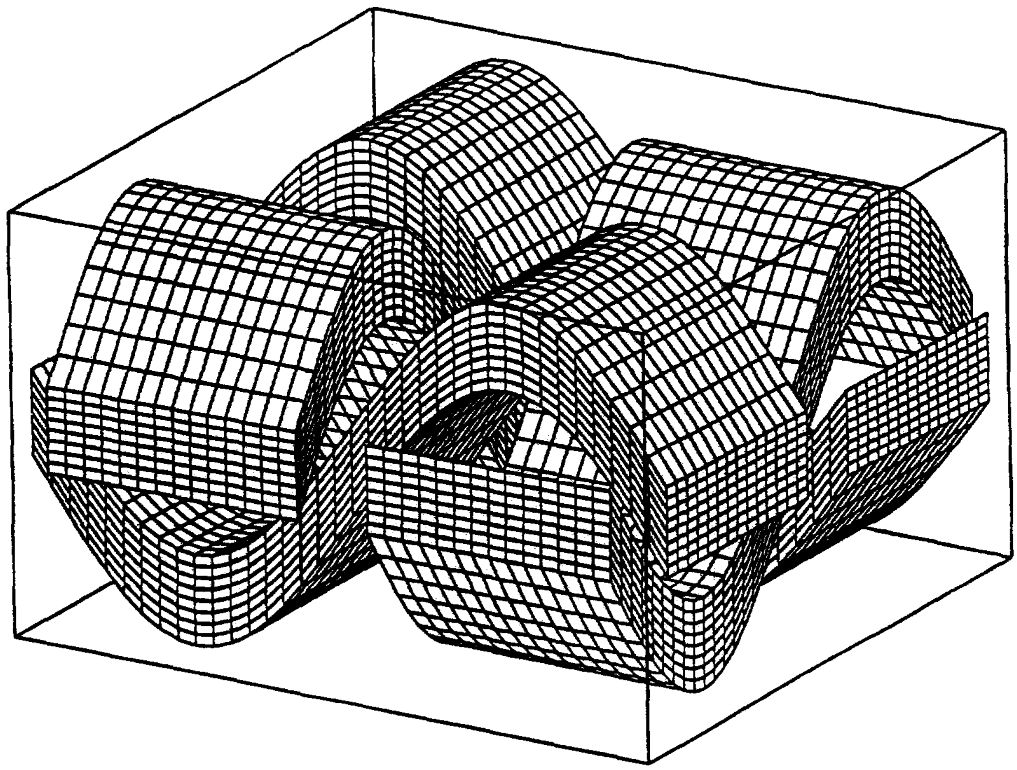


Fig. 2. Unit cell of plain weave composite.

where each $f_i \in L^2(\Omega)$, and $C_{ijkl}(y)$ is assumed to be a globally uniform (hence we drop the dependence on x), bounded measurable function, periodic in y (i.e. $C_{ijkl} \in L^{\infty}_{\text{per}}(Y)$), satisfying eqn (6) and

$$C_{ijkl}(y) = C_{jilk}(y) = C_{kjil}(y) \tag{10a}$$

$$\alpha s_{ij}s_{ij} \leqslant C_{ijkl}(y)s_{ij}s_{kl} \leqslant \beta s_{ij}s_{ij} \tag{10b}$$

for all second rank symmetric tensors s and some $\alpha > 0$ and $\beta > 0$. We assume the matrix to be a homogeneous isotropic body and so

$$C_{ijkl}(y) = \lambda_m \delta_{ik}\delta_{jl} + \mu_m \delta_{ij}\delta_{kl} + \mu_m \delta_{il}\delta_{kj}$$

for all y in the matrix portion of the unit cell. Here λ_m and μ_m are the Lamé constants for the matrix and $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$. Since the yarn is not isotropic, the properties of the yarn are computed by using the same homogenization procedure over a two-dimensional unit cell in the cross-section of the fiber bundle. We use the following material properties for the matrix and fiber:

Properties	Fiber	Matrix
E	10	1
ν	0.25	0.3

We now consider the variational form of eqn (9). We have $u^{\varepsilon} \in [H^1_0(\Omega)]^3$

$$\int_{\Omega} C_{ijkl}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon}_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x = \int_{\Omega} f_i v_i\,\mathrm{d}x \quad \forall v \in [H^1_0(\Omega)]^3. \tag{11}$$

Hence by choosing $v_i = u^{\varepsilon}_i$ in eqn (11) and using Korn's inequality with eqn (10), we see that u^{ε} is bounded in $[H^1_0(\Omega)]^3$ by a constant which does not depend on ε, and so u^{ε} converges weakly to a limit $u^{(0)}$ along some subsequence. Therefore, along this subsequence, the results of Lemma 1 hold, and we can apply Theorem 3 to the system (9) to obtain the two-scale homogenization.

At this point it is worthwhile to make a few remarks. First, the choice of boundary conditions (9b) was made only to adhere to the usual choice of the model homogenization problem in the literature. We could have just as easily let

$$u^{\varepsilon} = g_1 \quad \text{on} \quad \Gamma_u$$

$$C_{ijkl}\frac{\partial u^{\varepsilon}_k}{\partial x_l}n_j = g_2 \quad \text{on} \quad \Gamma_{\sigma}$$

where n_j is the unit normal on $\partial\Omega$, and Γ_u and Γ_{σ} are disjoint subsets of $\partial\Omega$ with positive Lebesgue measure such that $\bar{\Gamma}_u \cup \bar{\Gamma}_{\sigma} = \partial\Omega$, where we consider $g_1 \in H^{1/2}(\Gamma_u)$ and $g_2 \in L^2(\Gamma_{\sigma})$. In the same manner we can again show that u^{ε} is bounded in $[H^1(\Omega)]^3$ and then proceed in a similar fashion.

Second, it is clear from eqn (7) that the two-scale homogenized model has the same form as the original equation, and so the existence and uniqueness of the solution can certainly be expected in situations where it exists for the original system. Just as importantly, however, we see that the same numerical algorithms (e.g. finite elements) used to solve systems of the original type, can be used to solve the homogenized equations. Nevertheless, the two-scale system has twice as many equations to solve as the original system so that one should always consider if there are any advantages to decoupling the equations using eqn (8). This, of course, depends on the objective of the homogenization. If the goal is simply to calculate the effective moduli of the composite material, then essentially twice the necessary computing would be performed by solving eqn (7). In this instance, and in the case where a multitude of different boundary conditions on $\partial\Omega$ are to be studied, it becomes desirable to decouple the microstructure problem from the global equation.

By doing so, we obtain the following classical homogenization of eqn (9) :†

$$-\frac{\partial}{\partial y_i}\left[C_{ijkl}(y)\frac{\partial \chi_k^{mn}}{\partial y_l}\right] = \frac{\partial C_{ijmn}}{\partial y_i} \quad \text{in} \quad Y \tag{12a}$$

$$\left[\int_Y C_{ijkl}(y) - C_{ijmn}\frac{\partial \chi_m^{kl}}{\partial y_n}\,\mathrm{d}y\right]\frac{\partial^2 u_k^{(0)}}{\partial x_j \partial x_l} = f_i \quad \text{in} \quad \Omega \tag{12b}$$

$$u^{(0)} = (u_1^{(0)}, u_2^{(0)}, u_3^{(0)}) = 0 \quad \text{on} \quad \partial\Omega \tag{12c}$$

$$\chi_k^{mn}(x,\cdot)1-\text{periodic.} \tag{12d}$$

$\chi_k^{mn}(x,y)$ is the unique solution of eqns (34a) and (34d) in the space $C(\bar{\Omega}, H^1_{\text{per}}(Y)/\mathbb{R})$. We note that, since we are considering a uniformly periodic structure, χ is actually a constant with respect to the global variable x so that we write $\chi_k^{mn} \in H^1_{\text{per}}(Y)/\mathbb{R}$. Hence, by eqn (8) with $u_i^{(1)} = (\partial u_j^{(0)}/\partial x_k)(x)\chi_k^{ij}(y)$, we see that if f is continuous, $u_k^{(1)}$, $\partial u_k^{(1)}/\partial x_j$ and $\partial u_k^{(1)}/\partial y_j$ all satisfy eqn (2), and so by using the positive definiteness of C_{ijkl} and two-scale convergence we obtain the following strong convergence:

$$u_k^{\varepsilon} \to u_k^{(0)} + \varepsilon u_k^{(1)} \quad \text{strongly in} \quad H_0^1(\Omega) \quad \text{for} \quad k = 1,2,3. \tag{13}$$

We write eqn (12b) as

$$C_{ijkl}^{\text{eff}}\frac{\partial^2 u_k^{(0)}}{\partial x_j \partial x_l} = f_i \quad \text{in} \quad \Omega \tag{14}$$

where C_{ijkl}^{eff} represents the tensor of effective moduli for the homogenized composite. We note that our homogenization procedure has yielded a partial differential equation with constant coefficients whose solution is now quite easily obtainable through standard finite element techniques. Moreover, from eqn (13) we see that the displacements u^{ε} of eqn (9) converge strongly to the displacements $u^{(0)}$ of eqn (14), and the stresses $C\nabla u^{\varepsilon}$ converge strongly to the stresses $C^{\text{eff}}\nabla u^0$ of eqn (14).

We remark at this point that there have been some other attempts at homogenization of woven plain weave composites, as for example in Pandey and Hahn (1992) and Dasgupta and Bhandarkar (1994). The analysis in Pandey and Hahn (1992) is essentially based upon the assumption that the strains in the fiber and matrix are identical so that the computed effective moduli are the (Reuss) upper bounds. Although somewhat satisfactory as a first approximation, this scheme does not take into account the interaction between the fiber and the matrix in an adequate manner, ignoring the truly three-dimensional nature of the problem. In particular, no rigorous justification has been given in Pandey and Hahn (1992) that the solution to the equations of elasticity with coefficients so obtained, are in any sense the limit to the sequence of solutions of the original equations.

In the analysis of Dasgupta and Bhandarkar (1994), which is ostensibly based on the well known method of multiple scales, the effective moduli are actually calculated by applying "a uniform strain field" and "ensuring that the strain energy contributions from the dominant deformation mode is at least two orders of magnitude larger than the contributions from all other modes." Clearly such approximations cannot arise from the well developed theory of multi-scale asymptotics and therefore, an assessment of the accuracy of these results based on such approximations is difficult if not altogether impossible.

† Differentiation must be considered in the distributional sense unless sufficient continuity exists for C_{ijkl}, χ_i^{mn} and u_k.

It thus appears that our scheme, which entails the solution of eqn (12a) over the unit cell with eqn (12d) applied on the boundaries, is the first attempt at homogenization of plain weaves wherein the effect of microstructural detail has been correctly taken into account. The key issue to be addressed, then, is the development of a systematic approach to the solutions χ of the unit cell problem. One of the significant difficulties in solving such a problem numerically arises from the complexity of the composite geometry, and surfaces in the generation of the microstructural mesh. Although numerous schemes exist to generate such homogenization cells (George, 1991), we solve a system of partial differential equations to continuously deform the mesh consisting purely of eight-node hexahedral elements into the desired form. Specifically, we discretize the domain $[0, 1]^3$ and then assign displacement boundary conditions to certain nodes belonging to elements which comprise the fiber, as well as those on the boundaries. By doing so, we automate the mesh generation process, and so are able to perform various parametric studies using the finite element scheme which we have developed.

For this purpose, we write eqn (12a) in its variational form: $\chi_k^{mn} \in H^1_{\text{per}}(Y)/\mathbb{R}$ satisfies

$$\int_Y C_{ijkl}(y)\frac{\partial \chi_k^{mn}}{\partial y_l}\frac{\partial \phi_j^{mn}}{\partial y_i}\,\mathrm{d}y = \int_Y C_{ijmn}(y)\frac{\partial \phi_j^{mn}}{\partial y_i}\,\mathrm{d}y \quad \forall\, \phi_j^{mn} \in H^1_{\text{per}}(Y)/\mathbb{R}. \tag{15}$$

It is clear from eqn (10a) and the right-hand side of eqn (15) that χ_j^{mn} is symmetric in m and n. Thus, there are only eighteen independent components of χ, requiring the solution of just six separate boundary value problems corresponding to the six symmetric pairings of m and n. We devote the next section to the description of the results based on the process just outlined.

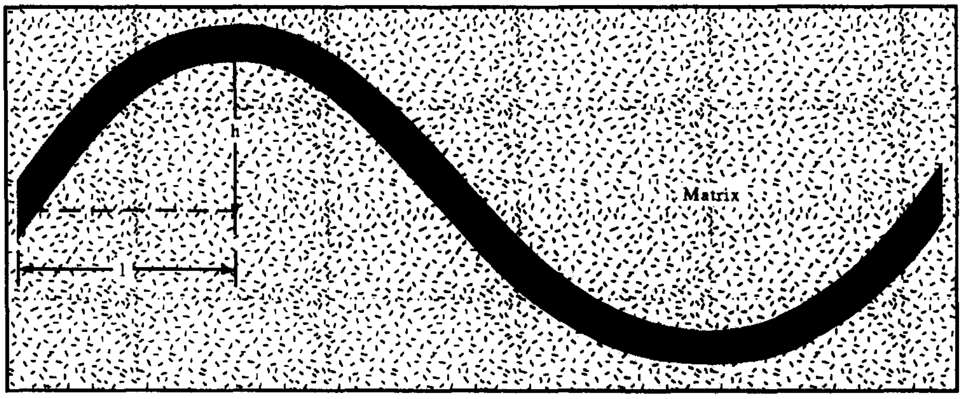
4. RESULTS AND PARAMETRIC STUDIES

The plain weave yarn architecture is such that each fill yarn passes alternately over and under every warp yarn as shown in Fig. 1 (*warp* refers to the long axis of the fabric roll whereas *fill* designates the roll width direction). Thus, while the yarn projections onto a flat surface form an orthogonal grid, the yarns are not unidirectional; rather, they exhibit periodic undulations about the fabric mid-plane. Under certain conditions, the geometry of the latter can significantly reduce some elements of the composite stiffness matrix. Consequently, an understanding of this phenomenon is important for design/analysis purposes.

Engineering studies in the literature have mainly focused on delineating the effect of fiber volume fraction on the composite stiffness. For a weave such as the one studied here, however, there are a number of other geometrical parameters whose effect on the stiffness must be systematically investigated for a fixed volume fraction. In particular, the change in the angle of undulation as well as changes in the cross-sectional area play a significant role in determining the overall effective properties of the composite material. Here, we study only the effect of fiber undulations and leave the study of the influence of cross-sectional area changes for future investigations.

To illustrate the affect of yarn undulations on global properties, we consider the idealized case wherein the undulations assume the form of a sine wave (Fig. 3) with amplitude h and quarter-period l. For this geometry and the representative properties, the FEM representation of Section 3 furnishes the results depicted in Figs 4(a–c). This shows the variation (decrease) of extensional stiffness C_{11} with weave angle (see Fig. 3) for a warp and fill yarn volume fraction of 30%. A significant dependence of extensional stiffness on weave angle is revealed by this figure.

While all elements of the stiffness matrix can be expected to vary with respect to the weave angle, not all such elements decrease. For example, while C_{12} decreases, C_{13} increases (see Fig. 4).



Weave angle = $\tan^{-1}(h/l)$

Fig. 3. Weave angle in undulating fiber.

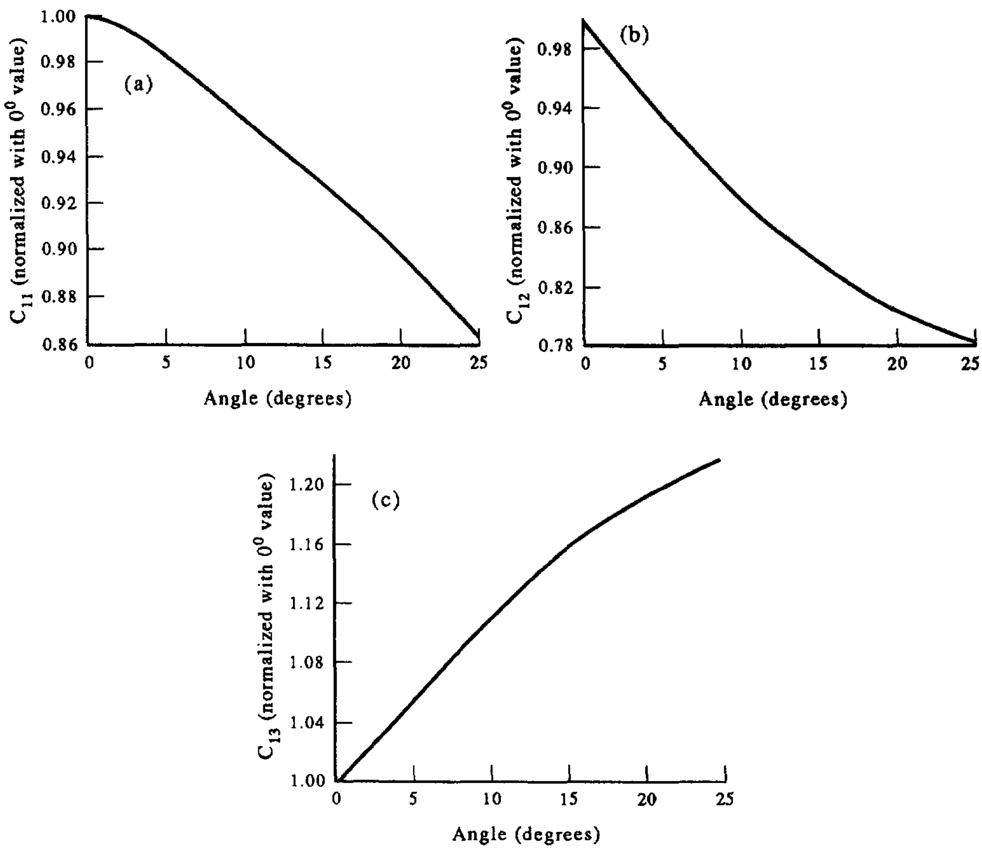


Fig. 4. Effect of weave angle on composite stiffness (plain weave fabric 30% fill and warp).

As was previously noted, the yarn geometry considered above is idealized in the sense of being smooth. Geometry which reflects less smooth phenomena can be examined using the method outlined in this paper once the period cell FEM mesh is defined.

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